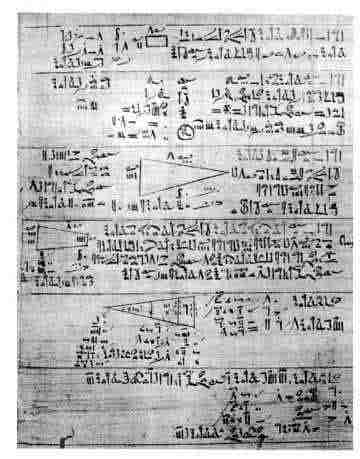
**Mathematics in Egyptian Papyri**

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| [Egyptian index](http://www-groups.dcs.st-and.ac.uk/history/Indexes/Egyptians.html) | [History Topics Index](http://www-groups.dcs.st-and.ac.uk/history/Indexes/HistoryTopics.html) |

[Version for printing](http://www-groups.dcs.st-and.ac.uk/history/PrintHT/Egyptian_papyri.html)

In the article [An overview of Egyptian mathematics](http://www-groups.dcs.st-and.ac.uk/history/HistTopics/Egyptian_mathematics.html) we looked at some details of the major Egyptian papyri which have survived. In this article we take a detailed look at the mathematics contained in them.



**The Rhind papyrus**

[Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html), in the Rhind papyrus, illustrates the Egyptian method of multiplication in the following way. Assume that we want to multiply 41 by 59. Take 59 and add it to itself, then add the answer to itself and continue:-

41 59

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

1 59

2 118

4 236

8 472

16 944

32 1888

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Since 64 > 41, there is no need to go beyond the 32 entry. Now go through a number of subtractions

41 - 32 = 9, 9 - 8 = 1, 1 - 1 = 0

to see that 41 = 32 + 8 + 1. Next check the numbers in the right hand column corresponding to 32, 8, 1 and add them.

41 59

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

1 59 tick

2 118

4 236

8 472 tick

16 944

32 1888 tick

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

2419

Notice that the multiplication is achieved with only additions, notice also that this is a very early use of binary arithmetic (see below). Reversing the factors we have:

59 41

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

1 41 tick

2 82 tick

4 164

8 328 tick

16 656 tick

32 1312 tick

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

2419

Notice that for this method to work we need to know that ever number is the sum of powers of 2. The ancient Egyptians would not have had a proof of this, nor would have appreciated that a proof was necessary. They would just know from practical experience that it could always be done. Basically we can think of the method as writing one of the numbers to base 2. In the examples above we have written

41 = 1.20 + 0.21 + 0.22 + 1.23 + 0.24 + 1.25

and

59 = 1.20 + 1.21 + 0.22 + 1.23 + 1.24 + 1.25.

Division works also using doubling. For example to divide 1495 by 65 we proceed as follows:

1 65

2 130

4 260

8 520

16 1040

We stop at this point because the next doubling will take us beyond 1495. Now we look for numbers in the right hand column which add up to 1495. We see that 1040 + 260 + 130 + 65 = 1495 and we tick the rows in which these numbers occur:

1 65 tick

2 130 tick

4 260 tick

8 520

16 1040 tick

Now add the numbers in the left hand column which are in ticked rows:

16 + 4 + 2 + 1 = 23,

so 1495 divided by 65 is 23.

What happens if the numbers do not divide exactly? Then the Egyptian method will yield fractions as the following example shows.

To divide 1500 by 65 proceed as before:

1 65

2 130

4 260

8 520

16 1040

Again we stop since the next doubling takes us beyond 1500. Now look for the numbers in the right hand column which add to a number n with 1500-65 < *n* ≤ 1500. [The Egyptians knew that this was always possible: can you prove that this is so?] In this case we have

1040 + 260 + 130 + 65 = 1495

and we are 5 short of our sum. Again tick the rows with these entries:

1 65 tick

2 130 tick

4 260 tick

8 520

16 1040 tick

Now add the numbers in the left hand column which are in ticked rows:

16 + 4 + 2 + 1 = 23,

so 1500 divided by 65 is 23 and 5/65 = 1/13 remaining. Hence the answer is 23 1/13.

We have cheated a little here for the fraction obtained is a unit fraction, that is a number of the form 1/n for n an integer. In fact the Egyptians only had fractions of this type and if the answer had not involved a unit fraction then the Egyptians would have written the fractional part as the sum of unit fractions. We see below how this was done but we examine a more general case.

The next problem is how to multiply and divide numbers involving fractions. The first important point is that the Egyptians only used unit fractions, and to be able to calculate a table was needed to convert twice a unit fraction into a sum of unit fractions. Now it might be supposed that doubling the unit fraction 1/5 would be easy and yield the sum of the unit fractions 1/5 + 1/5. However, for reasons which we do fully understand, this was not their approach. They wrote twice a unit fraction as the sum of distinct unit fractions. For example twice1/5 would be written as 1/3 + 1/15.

The Rhind papyrus gives a table for doubling unit fractions 1/*n* for *n* odd, *n* between 5 and 101. Note that [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) did not need to give the double of 1/*n* for *n* even since it is just 1/*m* where *m* = 2*n*. The doubling table for unit fractions begins

Unit fraction Double unit fraction

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

1/5 1/3 + 1/15

1/7 1/4 + 1/28

1/9 1/6 + 1/18

1/11

1/13

1/15 1/10 + 1/30

1/17 1/12 + 1/51 + 1/68

.... ..................

It is remarkable that there are no errors in the table. Certainly [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) would have been expert at calculating and this would not have been simply a copying exercise for him. There are few errors in the Rhind papyrus but those which there are appear to be errors of calculation, not of copying, since the incorrect result is carried forward rather than a return to the correct path which would happen from an error in copying.

There is the fascinating question of how these decompositions were found, and why some decompositions were chosen in preference to others. This is discussed in [[6](javascript:ref('%20%20%20R%20J%20Gillings,%20%3ci%3eMathematics%20in%20the%20Time%20of%20the%20Pharaohs%3c/i%3e%20%20%20(Cambridge,%20MA.,%201982).',6))] and further ideas, adding and correcting information from [[6](javascript:ref('%20%20%20R%20J%20Gillings,%20%3ci%3eMathematics%20in%20the%20Time%20of%20the%20Pharaohs%3c/i%3e%20%20%20(Cambridge,%20MA.,%201982).',6))], is given in [[17](javascript:ref('%20%20E%20M%20Bruins,%20Egyptian%20arithmetic,%20%20%3ci%3eJanus%3c/i%3e%20%20%3cb%3e68%3c/b%3e%20(1-3)%20(1981),%2033-52.%20%20',17))], [[18](javascript:ref('%20%20E%20M%20Bruins,%20Reducible%20and%20trivial%20decompositions%20concerning%20Egyptian%20arithmetics,%20%3ci%3eJanus%3c/i%3e%20%20%3cb%3e68%3c/b%3e%20(4)%20(1981),%20281-297.%20',18))], [[29](javascript:ref('%20%20R%20J%20Gillings,%20The%20Egyptian%20Mathematical%20Leather%20Role%20-%20line%208%20:%20How%20did%20the%20scribe%20do%20it?,%20%3ci%3eHistoria%20Math.%3c/i%3e%20%20%3cb%3e8%3c/b%3e%20(4)%20(1981),%20456-457.%20%20',29))] and [[35](javascript:ref('%20%20A%20E%20Raik,%20On%20the%20theory%20of%20Egyptian%20fractions%20(Russian),%20%3ci%3eIstor.-Mat.%20Issled.%20No.%3c/i%3e%20%20%3cb%3e23%3c/b%3e%20(1978),%20181-191;%20358.%20%20',35))]. The favourite rules which many historians such as Gillings believe guided the scribes in their choice of decomposition of 2/n into unit fractions are (1) prefer small numbers (2) the fewer terms the better, and never more than four (3) prefer even to odd numbers. However other historians such as Bruins argues against such rules. His argument is essentially that before applying these rules one would need to work out all unit decompositions of 2/*n* and there is no evidence that the Egyptians had any methods to do this.

As an example of how to use the table, let us examine Problem 21 of the Rhind papyrus. Note that 2/3 was an allowable Egyptian fraction despite not being a unit fraction.

Problem 21: Complete 2/3 and 1/15 to 1.

In modern terms, this asks for a fraction *x* such that

2/3 + 1/15 + *x* = 1.

The method of solution was to "get rid of" the fractions by multiplying through. In this case multiply each fraction by 15 to obtain

10 + 1 + *y* = 15.

This is called the "red auxiliary" equation since the scribe wrote this equation in red ink. [Of course it would not appear in this form but rather "complete 10 and 1 to 15".]

Now the answer to the red auxiliary equation is 4 so the original equation had solution twice × (twice × 1/15). From the doubling table we see that double 1/15 is 1/10 + 1/30. Doubling this gives 1/5 + 1/15 which is the required solution to Problem 21.

Another example of solving an equation is Problem 24 which asks:

Problem 24: A quantity added to a quarter of that quantity become 15. What is the quantity?

[Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) uses the "method of false position" which was still a standard method three thousand years later. In modern notation the problem is to solve

*x* + *x*/4 = 15.

[Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) guesses the answer *x* = 4. This is to remove the fraction in the *x*/4 term. Now with *x* = 4 the expression *x* + *x*/4 becomes 5. This is not the correct answer, for the expression is required to equal 15. However, 15 is 3 times 5 so taking 3 times his guess of *x* = 4, namely*x* = 12, gives [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) the correct result. Another interpretation, favoured by some historians, is that [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) thought of the method as dividing *x* into 4 equal pieces of a size to be determined. Now [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) computes *x* + *x*/4 getting 5 of these equal pieces. Each piece must now be three so that 5 pieces equals 15. Not very different to our previous way of thinking, but one which is likely to come closer to [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html)' way of thinking than our former description. Finally [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) checks his solution, or proves his answer is correct. He takes *x* = 4 × 3 = 12. Then *x*/4 = 3, so *x* + *x*/4 = 15 as required.

The methods of false position is used in Problems 24 to 29 of the Rhind Papyrus. However, in Problem 31 of the Papyrus [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html) uses the simpler method of pure division. This is discussed in detail in [[31](javascript:ref('%20%20M%20Guillemot,%20Les%20m%C3%A9thodes%20de%20simple%20fausse%20position%20dans%20les%20math%C3%A9matiques%20%C3%A9gyptiennes%20et%20les%20math%C3%A9matiques%20arabes,%20in%20%3ci%3eHistoire%20des%20math%C3%A9matiques%20arabes,%20Vol.%202%20Tipaza,%201990%3c/i%3e%20%20(Algiers,%201998),%20125-145.%20%20',31))].

Let us now see how to multiply, using Egyptian methods, 1 + 1/3 + 1/5 by 30 + 1/3.

1 1 + 1/3 + 1/5

2 2 + 2/3 + 1/3 + 1/15 = 3 + 1/15

4 6 + 1/10 + 1/30

8 12 + 1/5 + 1/15

16 24 + 1/3 + 1/15 + 1/10 + 1/30

2/3 2/3 + 1/6 + 1/18 + 1/10 + 1/30

1/3 1/3 + 1/12 + 1/36 + 1/20 + 1/60

Now here the row beginning 2/3 has been computed from 2/3 of 1 is 2/3, 2/3 of 1/3 is double 1/9 which is 1/6+1/18, 2/3 of 1/5 is double 1/15 which is 1/10 + 1/30.

Next find the numbers in the left hand column which add to 30+1/3. These are the rows marked with a tick:

1 1 + 1/3 + 1/5

2 3 + 1/15 tick

4 6 + 1/10 + 1/30 tick

8 12 + 1/5 + 1/15 tick

16 24 + 1/3 + 1/15 + 1/10 + 1/30 tick

2/3 2/3 + 1/6 + 1/18 + 1/10 + 1/30

1/3 1/3 + 1/12 + 1/36 + 1/20 + 1/60 tick

Add the entries in the right hand column of the rows which are ticked to get the result of the multiplication

46 + 1/5 + 1/10 + 1/12 + 1/15 + 1/30 + 1/36.

As a final look at the Rhind papyrus let us give the solution to Problem 50. A round field has diameter 9 khet. What is its area? Here is the solution as given by [Ahmes](http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Ahmes.html).

Take away 1/9 of the diameter, namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. Therefore it contains 64 setat of land.

Do it thus:

1 9

1/9 1

this taken away leaves 8

1 8

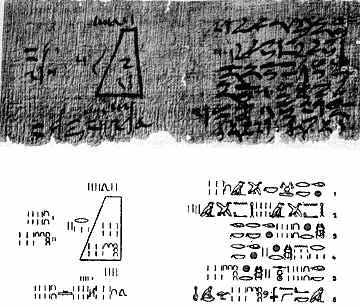
2 16

4 32

8 64

Its area is 64 setat.

Notice that the solution is equivalent to taking π = 4(8/9)2 = 3.1605. This is a remarkable result if one considers the date at which this approximation must have been discovered. The intriguing question is raised as to how such a discovery might have been made. Although we have no way of ever knowing this with certainty, several interesting conjectures have been suggested. In [[25](javascript:ref('%20%20P%20Gerdes,%20Three%20alternate%20methods%20of%20obtaining%20the%20ancient%20Egyptian%20formula%20for%20the%20area%20of%20a%20circle,%20%3ci%3eHistoria%20Math.%3c/i%3e%20%20%3cb%3e12%3c/b%3e%20(3)%20(1985),%20261-268.%20%20',25))] Gerdes gives three ideas which might have led the Egyptians to this result. Two such conjectures suggested in [[25](javascript:ref('%20%20P%20Gerdes,%20Three%20alternate%20methods%20of%20obtaining%20the%20ancient%20Egyptian%20formula%20for%20the%20area%20of%20a%20circle,%20%3ci%3eHistoria%20Math.%3c/i%3e%20%20%3cb%3e12%3c/b%3e%20(3)%20(1985),%20261-268.%20%20',25))] concern African crafts where a snake curve and a set of equidistant concentric rings are often seen. These two geometric designs are widespread in Africa and Gerdes shows how these could have led to a formula for the area of a circle. The third conjecture in [[25](javascript:ref('%20%20P%20Gerdes,%20Three%20alternate%20methods%20of%20obtaining%20the%20ancient%20Egyptian%20formula%20for%20the%20area%20of%20a%20circle,%20%3ci%3eHistoria%20Math.%3c/i%3e%20%20%3cb%3e12%3c/b%3e%20(3)%20(1985),%20261-268.%20%20',25))] relates to a board game "mancala" which was popular throughout Africa and ancient Egypt. The game involves comparing small circles with larger circles and may have provided the motivation for the area formula.



**The Moscow papyrus**

Although the mathematical methods we have described are found in various Egyptian documents, all the actual examples we have given so far have come from Rhind papyrus. Let us finish this article by looking at an example from the Moscow papyrus which many historians argue is the most impressive achievement of Egyptian mathematics. The problem is number 14 from the papyrus and it concerns the geometrical figure visible in the portion of Moscow papyrus seen in this image.

Example 14. Example of calculating a truncated pyramid. The base is a square of side 4 cubits, the top is a square of side 2 cubits and the height of the truncated pyramid is 6 cubits.

First we remark that by "calculate a pyramid" the author means "calculate the volume of the pyramid". Also not how appropriate this calculation is for the civilisation which today is universally known for the remarkable construction of pyramids.

The calculation begins by working out the area of the base: 4 × 4 = 16. Then the area of the top is worked out: 2 × 2 = 4. Next the product of the side of the base with the side of the top is computed: 4 × 2 = 8. These three are then added: 16 + 4 + 8 = 28. Now 1/3 of the height is taken, namely 2. Finally the product of 1/3 of the height with the previous sum of 28 is taken and the scribe writes:-

*Behold it is*56.

This example means that the Egyptian knew the formula for the volume (although of course not in the algebraic sense which we now think of formulas). If the base square has side *a*, the top square has side *b*, and the height is *h* then

*V* = *h* (*a*2 + *ab* + *b*2)/3.

[**References**](http://www-groups.dcs.st-and.ac.uk/history/HistTopics/References/Egyptian_papyri.html)**(43 books/articles)**

**Other Web sites:**

1. [Astroseti](http://www.astroseti.org/articulo/3659/) (A Spanish translation of this article)
2. [David Eppstein](http://www.ics.uci.edu/~eppstein/numth/egypt/) (Egyptian fractions)

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